

## TRANSLATING GRAPHS BY MEAN CURVATURE FLOW

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ABSTRACT. The aim of this work is studying translating graphs by mean curvature flow in  $\mathbb{R}^3$ . We prove non-existence of complete translating graphs over bounded domains in  $\mathbb{R}^2$ . Furthermore, we show that there are only three types of complete translating graphs in  $\mathbb{R}^3$ ; entire graphs, graphs between two vertical planes, and graphs in one side of a plane. In the last two types, graphs are asymptotic to planes next to their boundaries. We also prove stability of translating graphs and then we obtain a pointwise curvature bound for translating graphs in  $\mathbb{R}^3$ .

## 1. INTRODUCTION

Mean curvature flow evolves hypersurfaces in the unit normal direction with speed equal to the mean curvature at each point. It is the steepest descent flow for the area functional. In particular, minimal hypersurfaces are stationary solutions. In other words, a family of smoothly embedded hypersurfaces  $(\mathcal{M}_t)_{t \in I}$  moves by mean curvature if

$$(1.1) \quad \frac{\partial x}{\partial t} = \vec{H}(x),$$

for  $x \in \mathcal{M}_t$  and  $t \in I$ ,  $I \subset \mathbb{R}$  an open interval. Here  $\vec{H}(x)$  is the mean curvature vector at  $x \in \mathcal{M}_t$ .

The evolution equation (1.1) can develop singularities in finite time  $T$ , which are classified into two types according to the rate at which the maximal curvature,  $\max_{\mathcal{M}_t} |A(t)|$ , tends to infinity for  $t \rightarrow T$ . Here  $|A(t)|$  is the second fundamental form of  $\mathcal{M}_t$ . By proving the monotonicity formula, Huisken [11] showed that the flow is asymptotically self-similar near a given type-I singularity and, thus, is modeled by self-shrinking solutions of the flow. However, the examples of convergence in [1, 2] indicate that type-II singularities are modeled by translating surfaces. Also, Huisken and Sinestrari [13] proved that if the initial surface  $\mathcal{M}_0$  has nonnegative mean curvature, then any limiting flow of a type-II singularity has convex surfaces  $\widetilde{\mathcal{M}}_t$ ,  $t \in \mathbb{R}$ . Furthermore, either  $\widetilde{\mathcal{M}}_t$  is a strictly convex translating soliton or (up to rigid motion)  $\widetilde{\mathcal{M}}_t = \mathbb{R}^{n-k} \times \Sigma_t^k$ , where  $\Sigma_t^k$  is a lower dimensional strictly convex translating soliton in  $\mathbb{R}^{k+1}$ . The proof of this theorem used an important theorem of Hamilton [9], which states that any strictly convex eternal solution to the mean curvature flow, where the mean curvature assumes its maximum value at a point in space-time must be a translating solution.

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*Key words and phrases.* mean curvature flow, compactness theorem, translating graphs.

Note that solitons generally move by symmetries of Euclidean space, either scaling symmetries or translations, that means for these surfaces, we have

$$(1.2) \quad \frac{\partial x}{\partial t} = \vec{H}(x) = \vec{C} + \vec{v},$$

where  $\vec{C}$  is the velocity vector of the translation and  $\vec{v}$  is a vector field tangent to the surface  $\mathcal{M}_t$ . When  $\vec{C} = -\frac{x}{2}$  or  $\vec{C} = e_{n+1}$ , we will respectively get self similar shrinkers and vertically translating surfaces in  $\mathbb{R}^{n+1}$ . If  $\vec{C} = Ce_{n+1}$ , and taking the inner product with the unit normal vector  $v$ , we obtain the following equation for translating surfaces

$$(1.3) \quad H = C\langle e_{n+1}, v \rangle.$$

Translating graphs by mean curvature flow are translating surfaces that can be viewed as a graph of a function over a domain. Let the graph of the function  $u = u(x)$  be a translating graph by the mean curvature flow. Since  $H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ , the graph of  $u$  is a vertically translating graph with constant speed  $C$  if and only if  $u$  is a solution to the following equation

$$(1.4) \quad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{C}{\sqrt{1+|\nabla u|^2}}.$$

In this work, we study translating graphs by mean curvature flow in  $\mathbb{R}^3$ . First of all, we prove these graphs are stable minimal surfaces in a certain conformal metric. Therefore, the curvature estimate for minimal surfaces gives us a curvature estimate for translating graphs. Using this curvature estimate, we obtain non-existence of complete translating graphs over bounded domains in  $\mathbb{R}^2$ . Furthermore, we show that there are only three types of complete translating graphs in  $\mathbb{R}^3$ ; entire graphs, graphs between two vertical planes, and graphs in one side of a plane. In the last two types, graphs are asymptotic to planes next to their boundaries.

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## 2. STABILITY OF TRANSLATING GRAPHS

Colding and Minicozzi proved self similar shrinkers are stable minimal surfaces in a certain conformal metric [5, 6]. In this section, using the same method, we prove the similar statement for translating graphs, i.e. translating graphs are stable minimal hypersurfaces in a conformal metric.

From now on without loss of generality we assume that the speed  $C = -1$ . Define the functional  $F$  on a hypersurface  $\Sigma \subset \mathbb{R}^3$  by

$$(2.1) \quad F(\Sigma) = \int_{\Sigma} e^{x_3} d\mu.$$

In the following lemma we prove translating surfaces are stable minimal hypersurfaces in  $\mathbb{R}^3$  with respect to the conformal metric  $g_{ij} = e^{x_3}\delta_{ij}$ .

**Lemma 2.1.** *If  $x' = \beta v$  is a compactly supported normal variation of a hypersurface  $\Sigma \subset \mathbb{R}^3$  and  $s$  is the variation parameter, then*

$$\frac{\partial}{\partial s} F(\Sigma_s) = \int_{\Sigma} \beta (H + \langle e_3, v \rangle) d\mu.$$

*Proof.* By the first variation formula we have  $(d\mu)' = \beta H d\mu$ . Also the  $s$  derivative of  $\log(e^{x_3})$  is given by  $\langle e_3, v \rangle$ . Thus we have the lemma.  $\square$

Let the graph  $\tilde{\Sigma}$  be the graph  $\Sigma$  respect to the conformal metric  $g$ . We know that the hypersurface  $\tilde{\Sigma}$  subset of Riemannian manifold  $(\mathbb{R}^3, g)$  is stable minimal hypersurface if and only if for every smooth compact support function  $\eta$  over  $\tilde{\Sigma}$  we get

$$(2.2) \quad \int_{\tilde{\Sigma}} \langle \tilde{L}\eta, \eta \rangle \leq 0,$$

where

$$(2.3) \quad \tilde{L}\eta = \tilde{\Delta}\eta + |\tilde{A}|^2\eta + \widetilde{Ric}_{\mathbb{R}^3}(v, v)\eta.$$

Note that

$$(2.4) \quad \widetilde{Ric}_{\mathbb{R}^3}(v, v) = 0.$$

Also we have

$$(2.5) \quad \tilde{\Delta}\eta = e^{-x_3}(\Delta\eta - \nabla^k x_3 \nabla_k \eta) = e^{-x_3}(\Delta\eta + \langle e_3, \nabla\eta \rangle),$$

and the norm of second fundamental form respect to the conformal metric  $g$  is

$$(2.6) \quad |\tilde{A}|^2 = e^{-x_3}(|A|^2 - \frac{H^2}{2}),$$

because  $H = \langle e_3, v \rangle$  and

$$(2.7) \quad \begin{aligned} \tilde{h}_{ij} &= e^{-\frac{x_3}{2}} h_{ij} - e^{-x_3} \frac{\partial}{\partial v} e^{\frac{x_3}{2}} \delta_{ij} \\ &= e^{-\frac{x_3}{2}} \left( h_{ij} - \frac{H}{2} \delta_{ij} \right), \end{aligned}$$

where  $\tilde{v}$  is the upward unit normal of  $\tilde{\Sigma}$ .

Now we are defining the second order operator  $L$  by

$$Lu = \Delta u + |A|^2 u + \langle e_3, \nabla u \rangle.$$

**Definition 2.2.** We say a translating surface is  $L$ -stable, if for any compactly supported function  $\eta$  we have

$$(2.8) \quad \int_{\Sigma} \eta L\eta e^{x_3} \leq 0.$$

The linear operator  $L$  is associated to normal perturbations of  $H + \langle e_3, v \rangle$ . The function  $H + \langle e_3, v \rangle$  is invariant under translations in  $\mathbb{R}^2$ , therefore  $\langle \mathbf{v}, v \rangle$  is in the kernel of  $L$  for any constant vector  $\mathbf{v}$ .

**Proposition 2.3.** *If the translating graph  $\Sigma$  in  $\mathbb{R}^3$  with respect to the Euclidean metric is  $L$ -stable, then it is a stable minimal hypersurface in  $\mathbb{R}^3$  with respect to the conformal metric  $g_{ij} = e^{x_3} \delta_{ij}$ .*

*Proof.* Let  $\eta$  be a smooth compactly supported function over  $\Sigma$  equations (2.3), (2.4), (2.5) and (2.6) gives

$$(2.9) \quad \tilde{L}\eta = e^{-x_3} \left( \Delta\eta + |A|^2\eta - \frac{H^2}{2}\eta + \langle e_3, \nabla\eta \rangle \right).$$

This implies

$$\begin{aligned}
\int_{\tilde{\Sigma}} \langle \tilde{L}\eta, \eta \rangle &= \int_{\Sigma} \eta \tilde{L}\eta e^{2x_3} \\
&= \int_{\Sigma} e^{x_3} \left( \eta \Delta \eta + |A|^2 \eta^2 - \frac{H^2}{2} \eta^2 + \eta \langle e_3, \nabla \eta \rangle \right) \\
&\leq \int_{\Sigma} e^{x_3} \eta (\Delta \eta + |A|^2 \eta + \langle e_3, \nabla \eta \rangle) \\
(2.10) \quad &= \int_{\Sigma} \eta L\eta e^{x_3}.
\end{aligned}$$

Now the result follows from (2.10), the definition of stability for minimal surfaces (2.2) and definition of  $L$ -stability (2.8).  $\square$

**Lemma 2.4.** *For every constant vector  $\mathbf{v}$ , we have  $L\langle \mathbf{v}, v \rangle = 0$ .*

*Proof.* We give a computational proof. Let  $\gamma_i$  be an orthonormal frame for  $\Sigma$  and set  $\xi = \langle \mathbf{v}, v \rangle$ . Working at a fixed point  $P$  and choosing the frame  $\gamma_i$ , so that  $\nabla_{\gamma_i}^T \gamma_j(P) = 0$ , differentiating gives at  $P$  that

$$(2.11) \quad \nabla_{\gamma_i} \xi = \langle \mathbf{v}, \nabla_{\gamma_i} v \rangle = -a_{ij} \langle \mathbf{v}, \gamma_j \rangle.$$

Using Codazzi equation at  $P$ , we have

$$\nabla_{\gamma_k} \nabla_{\gamma_i} \xi = -a_{ik,j} \langle \mathbf{v}, \gamma_j \rangle - a_{ij} \langle \mathbf{v}, a_{jk} v \rangle.$$

Taking the trace gives

$$(2.12) \quad \Delta \xi = \langle \mathbf{v}, \nabla H \rangle - |A|^2 \xi.$$

Notice that

$$\begin{aligned}
\nabla_{\gamma_i} H &= -\nabla_{\gamma_i} \langle e_3, v \rangle \\
&= -\langle \nabla_{\gamma_i} e_3, v \rangle - \langle e_3, \nabla_{\gamma_i} v \rangle \\
&= a_{ij} \langle e_3, \gamma_j \rangle.
\end{aligned}$$

Therefore, using (2.11) we have

$$(2.13) \quad \langle \nabla H, \mathbf{v} \rangle = a_{ij} \langle e_3, \gamma_j \rangle \langle \gamma_i, \mathbf{v} \rangle = -\langle e_3, \nabla \xi \rangle.$$

Thus, by (2.12) and (2.13) we get

$$L\xi = \Delta \xi + \langle \nabla \xi, e_3 \rangle + |A|^2 \xi = 0. \quad \square$$

**Theorem 2.5.** *Translating graphs in  $\mathbb{R}^3$  are  $L$ -stable.*

*Proof.* Since  $\Sigma$  is a graph, there is a unit vector  $\mathbf{v}$  in  $\mathbb{R}^3$  so that  $\langle \mathbf{v}, v(x) \rangle \neq 0$  for all  $x \in \Sigma$ . We define the function  $\xi$  on  $\Sigma$  by

$$\xi(x) = \langle \mathbf{v}, v(x) \rangle.$$

It follows that  $0 < \xi \leq 1$  and, by Lemma 2.4, that  $L\xi = 0$ . Given any smooth compactly supported function  $\eta$  on  $\Sigma$ , the function  $\phi = \eta\xi$  satisfies

$$\begin{aligned}
L(\phi) &= \eta L\xi + \xi (\Delta \eta + \langle e_3, \nabla \eta \rangle) + 2\langle \nabla \eta, \nabla \xi \rangle, \\
(2.14) \quad &= \xi (\Delta \eta + \langle e_3, \nabla \eta \rangle) + 2\langle \nabla \eta, \nabla \xi \rangle.
\end{aligned}$$

Using Stokes' theorem with  $\frac{1}{2} \operatorname{div} (\xi^2 \nabla \eta^2 e^{x_3})$ , we obtain

$$(2.15) \quad \int \frac{1}{2} \langle \nabla \eta^2, \nabla \xi^2 \rangle e^{x_3} = - \int \xi^2 (\eta \Delta \eta + |\nabla \eta|^2 + \eta \langle e_3, \nabla \eta \rangle) e^{x_3}$$

Applying (2.14) and (2.15), we obtain

$$(2.16) \quad \int \phi L(\phi) e^{x_3} = - \int \xi^2 |\nabla \eta|^2 e^{x_3} \leq 0 \quad \square$$

When  $\Sigma$  is graphical, we have a direction  $\omega$  for which  $\xi > 0$  for all  $x \in \Sigma$ . Given a smooth compactly supported function  $\phi$ , we take  $\eta(x) := \phi(x)/\xi(x)$ . This means that (2.16) is true for any compactly supported function  $\phi$ , which is the definition of  $L$ -stability.

*Remark 2.6.* Since translating graphs are  $L$ -stable, compactness theorem for stable minimal graphs in  $\mathbb{M}^3$ , where  $\mathbb{M}^3$  is a three dimensional Riemannian manifold, implies compactness theorem for translating graphs in  $\mathbb{R}^3$  with polynomial volume growth.

### 3. CURVATURE ESTIMATE

In this section, we obtain a point wise curvature estimate for translating graphs by mean curvature flow in  $\mathbb{R}^3$ . For reaching this goal, we state theorem 2.10 in [4], which is Schoen curvature estimate for two dimensional minimal hypersurfaces  $\Sigma$  immersed in Riemannian manifold  $\mathbb{M}^3$  with sectional curvature  $K_{\mathbb{M}}$  [15]. For  $x \in \mathbb{M}$ ,  $B_s(x)$  denotes the extrinsic geodesic ball with radius  $s$  and center  $x$ . Similarly, For  $x \in \Sigma$ ,  $B_s^\Sigma(x) \subset \Sigma$  denotes the intrinsic geodesic ball and  $r$  the intrinsic distance to  $x$ .

**Theorem 3.1** (Schoen Curvature estimate [15], Colding-Minicozzi [4]). *If  $\Sigma^2 \subset \mathbb{M}^3$  is an immersed stable minimal surface with trivial normal bundle and  $B_{r_0} = B_{r_0}^\Sigma(x) \subset \Sigma \setminus \partial\Sigma$ , where  $|K_{\mathbb{M}}| \leq k^2$  and  $r_0 < \rho_1(\pi/k, k)$  (with  $\rho_1 < \min\{\pi/k, k\}$ ), then for some  $C = C(k)$  and all  $0 < \sigma \leq r_0$ ,*

$$(3.1) \quad \sup_{B_{r_0-\sigma}^\Sigma} |A|^2 \leq C\sigma^{-2}.$$

For applying this theorem to obtain the curvature estimate, we need to compute the sectional curvature of  $\mathbb{R}^3$  respect to the conformal metric  $g$ . By doing some simple computations, we get for every  $1 \leq i, j \leq 2$ ,  $K_{ij} = -\frac{1}{4}e^{-x_3}$ , and  $K_{i3} = K_{3i} = 0$ .

**Theorem 3.2.** *Let  $\Sigma^2 \subset \mathbb{R}^3$  be a complete translating graph in mean curvature flow, if  $B_{r_0e}^\Sigma(p) \subset (\Sigma \cap B_1(p)) \setminus \partial(\Sigma \cap B_1(p))$ , and  $r_0e^{1/2} < \rho_1(\pi e^{-1}, e)$ , then for some  $C$  and all  $0 < \sigma \leq r_0$ ,*

$$(3.2) \quad \sup_{B_{r_0-\sigma}^\Sigma} |A|^2 \leq C\sigma^{-2}.$$

*Proof.* For point  $p \in \Sigma$ , let  $B_1$  be the Euclidean unit ball of radius 1 and center  $p$  in  $\mathbb{R}^3$ . Define  $\tilde{\Sigma} = \Sigma \cap B_1$ , note that  $\tilde{\Sigma}$  is immersed submanifold in  $B_1 = B_1(0)$  and  $|\hat{A}|(p) = |A|(p)$ . Now let  $\tilde{B}_1$  be  $B_1$  with respect to the metric  $g_{ij} = e^{x_3-p_3}\delta_{ij}$ . From theorem 2.5,  $\tilde{\Sigma}$  is stable minimal hypersurface in  $\tilde{B}_1$ . Note that we only multiplied the metric  $g_{ij}$  in the theorem by a constant  $e^{-p_3}$ , which wont change the results of theorem.

Let the distance  $d$  be the distance corresponding to Euclidean metric and  $\tilde{d}$  the distance corresponding to the conformal metric. Note that  $B_{r_0}^{\tilde{\Sigma}} = \{x \in \Sigma : \tilde{d}(x, p) < r_0\}$ , where  $\tilde{d}(x, p)$  is the infimum of the length of geodesic curves connecting two points  $p$  and  $x$  in  $\tilde{\Sigma}$ . For  $x = (x_1, x_2, x_3) \in \Sigma \cap B_1$ , define minimizing

geodesic  $\gamma : [0, 1] \rightarrow \Sigma \cap B_1$  in Euclidean metric, connecting  $p$  and  $x$  in  $\Sigma$ , such that  $\gamma(0) = p$  and  $\gamma(1) = x$ . Using Cauchy Schwartz inequality we have

$$\begin{aligned}
 \tilde{d}(x, 0) &\leq \int_0^1 \|\gamma'(t)\| dt \\
 &= \int_0^1 \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt \\
 &= \int_0^1 e^{\frac{\gamma_3(t) - p_3}{2}} |\gamma'(t)| dt \\
 &\leq \left( \int_0^1 e^{\gamma_3(t) - p_3} dt \right)^{\frac{1}{2}} \left( \int_0^1 \gamma'^2(t) dt \right)^{\frac{1}{2}} \\
 (3.3) \quad &= e^{1/2} d(x, 0).
 \end{aligned}$$

This implies if  $x \in B_r^\Sigma(p) \subset B_1$ , then  $x \in B_{re^{1/2}}^{\tilde{\Sigma}}(p)$ . Now define minimizing geodesic  $\tilde{\gamma} : [0, 1] \rightarrow \tilde{\Sigma} \cap \tilde{B}_1$  connecting  $p$  and  $x$  in  $\tilde{\Sigma} \cap \tilde{B}_1$ , such that  $\tilde{\gamma}(0) = p$  and  $\tilde{\gamma}(1) = x$ . Using Cauchy Schwartz inequality we have

$$\begin{aligned}
 e^{-1/2} d(x, 0) &\leq \int_0^1 e^{-1/2} |\tilde{\gamma}'(t)| dt \\
 &\leq \int_0^1 e^{(\tilde{\gamma}_3(t) - p_3)/2} |\tilde{\gamma}'(t)| dt \\
 &= \int_0^1 \sqrt{\langle \tilde{\gamma}'(t), \tilde{\gamma}'(t) \rangle} dt \\
 (3.4) \quad &= \int_0^1 \|\tilde{\gamma}'(t)\| dt = \tilde{d}(x, 0).
 \end{aligned}$$

This implies if  $x \in B_r^{\tilde{\Sigma}}(p) \subset \tilde{B}_1$ , then  $x \in B_{re^{1/2}}^\Sigma(p)$ . Note that if  $x \in B_{r_0e^{1/2}}^{\tilde{\Sigma}}(p)$ , then  $X \in B_{r_0e}^\Sigma(p) \subset (\Sigma \cap B_1(p)) \setminus \partial(\Sigma \cap B_1(p))$  which implies that  $x \in \hat{\Sigma} \setminus \partial\hat{\Sigma}$ . Also if  $x \in B_{r_0-\sigma}^\Sigma$ , then  $x \in B_{e^{1/2}(r_0-\sigma)}^{\tilde{\Sigma}}$ .

Since sectional curvature of  $\tilde{B}_1$  is bounded ( $|K_{\tilde{B}_1}| < e$ ), formula (2.6) and theorem 3.1 imply that for  $r_0e^{1/2} < \rho_1(\pi e^{-1}, e)$  and  $B_{r_0e^{1/2}}^{\tilde{\Sigma}}(p) \subset \hat{\Sigma} \setminus \partial\hat{\Sigma}$ , for some  $C$  we obtain for all  $x \in B_{r_0-\sigma}^\Sigma(p)$ ,

$$(3.5) \quad |A|^2(x) \leq e^{x_3 - p_3} |\tilde{A}|^2(x) \leq Ce^{(e^{1/2}\sigma)^{-2}} = C\sigma^{-2}.$$

□

#### 4. CLASSIFICATION OF COMPLETE TRANSLATING GRAPHS

In this section, we prove translating graphs over a domain  $\Omega \subset \mathbb{R}^2$  are asymptotic to a minimal surface next to the  $\partial\Omega$ . Which implies non existence of translating graphs over a bounded domain. Also, it concludes that complete translating graphs in  $\mathbb{R}^3$  can only be an entire graph over  $\mathbb{R}^2$  or be in one side of a vertical plane or between two vertical parallel planes.

**Lemma 4.1.** *If  $\Sigma \subset \mathbb{R}^3$  is a translating graph, then there is a  $\delta > 0$  such that for every  $p \in \Sigma$ ,  $\Sigma$  is a graph over the disk  $D_\delta(p) \subset T_p\Sigma$  of radius  $\delta$  centered at  $p$ .*

*Proof.* We will prove this lemma in two different ways.

First, we prove by contradiction, assume that there is a sequence of points  $p_n \in \Sigma$  so that  $\Sigma$  is a graph over the disk  $D_{\delta_n}(p_n) \subset T_{p_n}\Sigma$ , and  $\delta_n \rightarrow 0$ . For fix  $R > 0$ , we define  $f_n = B_R(p_n) \cap \Sigma$ . Now we translate each  $f_n$  so that  $p_n$  goes to the origin, we call this graph  $g_n$ . Each  $g_n$  is a translating graph in the ball  $B_R(0) \subset \mathbb{R}^3$ , we choose a subsequence of  $g_n$  so that the tangent planes of  $g_n$  at origin converges to some plane at origin. Compactness theorem for translating graphs implies that there is a subsequence of  $g_n$  which are converging to a translating surface  $g_\infty$ . For  $0 \in g_\infty$  there is a  $\delta > 0$  so that  $g_\infty$  is a graph over the disk  $D_\delta(0) \subset T_0g_\infty$ . Thus there is an  $N$  large so that  $g_n$  is a graph over the disk  $D_\delta(0) \subset T_0g_n$ , for every  $n > N$ .

Second way:

Let  $p$  and  $q$  be two different point in  $\Sigma$ . There is a geodesic  $\gamma : [0, 1] \rightarrow \Sigma$ , so that  $\gamma(0) = p$  and  $\gamma(1) = q$ . Now for  $v$  normal vector to the  $\Sigma$  we have

$$\begin{aligned} |v(p) - v(q)| &\leq \int_0^1 |\nabla_{\gamma'} v(\gamma(t))| dt \\ (4.1) \qquad \qquad &= \int_0^1 |A(\gamma(t))| dt. \end{aligned}$$

Hence the theorem 3.2 implies the lemma.  $\square$

**Theorem 4.2.** *There is no complete translating graph  $\Sigma \subset \mathbb{R}^3$  with nonzero constant speed  $C$  over a bounded connected domain  $\Omega \subset \mathbb{R}^2$  with smooth boundary.*

*Proof.* The proof inspired by the one used in [10]. Suppose  $\Sigma$  is a complete immersed translating graph over a domain  $\Omega \subset \mathbb{R}^2$ . Lemma 4.1 implies there exists  $\delta > 0$  such that for each  $p \in \Sigma$ ,  $\Sigma$  is a graph in exponential coordinates over the disk  $D_\delta(p) \subset T_p\Sigma$  of radius  $\delta$ , centered at the  $p$ . We denote this graph by  $G(p) \subset \Sigma$ , has bounded geometry. Note that  $\delta$  is independent of  $p$  and the bound on the geometry of  $G(p)$  is uniform as well.

We define  $F(p)$ ; the surface  $G(p)$  translated to height zero  $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ , i.e, let  $\alpha_p$  be the isometry of  $\mathbb{R}^3$  which takes  $p$  to  $\pi(p)$ , we denote  $F(p) = \alpha_p(\Sigma)$ .

Now, let  $p \in \Sigma$ , since  $\Sigma$  is a graph over  $\Omega$ , there is a function  $u : \Omega \rightarrow \mathbb{R}^3$  so that  $\Sigma$  is the graph of  $u$ . If  $\Sigma$  is not an entire graph then  $\partial\Omega \neq \emptyset$ . Since  $\Sigma$  is a translating graph by mean curvature flow,  $u$  has bounded gradient on relatively compact subsets of  $\Omega$ . Let  $q \in \partial\Omega$  be such that  $u$  does not extend to any neighborhood of  $q$ .

Let  $q_n$  be a sequence in  $\Omega$  converging to  $q$ , and let  $p_n = (q_n, u(q_n)) \in \Sigma$  be images of  $q_n$  in  $\Sigma$ . Let  $F_n$  denote the image of  $G(p_n)$  under the vertical translation taking  $p_n$  to  $q_n$ . Observe that  $T_{q_n}(F_n)$  converges to the vertical plane  $P$ , for any subsequence of the  $q_n$ . Otherwise the graph of bounded geometry  $G(p_n)$ , would extend to a vertical graph beyond  $q$ , for  $q_n$  close enough to  $q$ . Hence  $f$  would extend; a contradiction.

For  $q \in \Omega$ , we define  $L_\delta(q)$  a line of length  $2\delta$  centered at  $q$ . Let  $L_\delta(q)$  be the line whose normal vector has the same direction as the normal vector of limit normal vectors of  $F_n$ . Since each  $F_n$  is a graph over  $D_\delta(p_n) \subset T_{p_n}(F_n)$ , the surfaces  $F_n$  are bounded horizontal graphs over  $L_\delta(q) \times [-\delta, \delta]$  for  $n$  large. The compactness theorem for translating graphs implies that there is a subsequence of  $F_n$ 's which are converging to a translating surface  $F$ . The surface  $F$  is tangent to  $L_\delta(q) \times [-\delta, \delta]$  at  $q$ . Note that  $F = L_\delta(q) \times [-\delta, \delta]$ . Because if it is not the case there is a small

$\epsilon > 0$  so that  $F(q - \epsilon \vec{n}(q))$  has two positive and negative values, where  $\vec{n}$  is the unit normal to the  $L_\delta(q)$ . Therefore for  $n$  large  $F_n$  is not a graph, which is contradiction.

The plane  $P = L_\delta(q) \times [-\delta, \delta]$ , because both planes  $P$  and  $L_\delta(q) \times [-\delta, \delta]$  are passing through the point  $q \in \partial\Omega$  and their normal vectors are the same.

In this point, we prove  $u(q_n) \rightarrow +\infty$  or  $u(q_n) \rightarrow -\infty$ . Let  $l$  be a line of length  $\epsilon$  inside  $\Omega$ , starting at  $q$ , orthogonal to  $\partial\Omega$  at  $q$ . Let  $f$  be the graph of  $u$  over  $l$ . At points near  $q$ ,  $l$  has no horizontal tangents, because tangent planes of  $u$  at these points are converging to  $P$ . So we assume  $u$  is increasing along  $l$  as one converges to  $q$ . If  $u$  is bounded above, then  $f$  would have a finite limit point  $(q, l_q)$  and  $f$  would have finite length up till  $(q, l_q)$ . Since  $\Sigma$  is complete,  $(q, l_q) \in \Sigma$ , which contradicts by  $\Sigma$  has a vertical tangent plane at  $(q, l_q)$ .

Note that from Lemma 2.4,  $1/w$  satisfies an elliptic partial differential equation. Thus by the Harnack inequality, for any sequence  $q_n \in \Omega$  converging to  $q$  we have  $w(q_n) \rightarrow +\infty$ . That means  $H(q_n) \rightarrow 0$ .

Which is contradiction, since the domain is bounded, the mean curvature of the graph next to the boundary should converge to the mean curvature of the cylinder  $\partial\Omega \times \mathbb{R}$ , which is not zero.  $\square$

**Corollary 4.3.** *If  $\Sigma$  is a complete translating graph over a domain  $\Omega \subset \mathbb{R}^2$ , then next to the boundary of  $\Omega$ ,  $\Sigma$  is asymptotic to a plane. So a translating graph over  $\mathbb{R}^2$  can only be between 2 parallel planes or in one side of a plane or an entire graph.*

*Proof.* From the proof of Theorem 4.2, next to the boundary of  $\Omega$ , the graph  $\Sigma$  converges to a minimal surface. Since  $\Sigma$  is complete, it can only converge to a vertical plane.  $\square$

## REFERENCES

- [1] Angenent, S. B., Velazquez, J. J. L.: Asymptotic shape of cusp singularities in curve shortening. Duke Math. J. 77, no. 1, 71-110 (1995).
- [2] Angenent, S. B., Velazquez, J. J. L.: Degenerate neckpinches in mean curvature flow. J. Reine Angew. Math. 482, 15-66 (1997).
- [3] Choi, H. I., Schoen, R.: The space of minimal embeddings of a surface into a three dimensional manifold of positive Ricci curvature, Invent. Math. 81, no. 3, 387-394 (1985).
- [4] Colding, T. H., Minicozzi II, W. P.: A course in minimal surfaces, Graduate Studies in Mathematics, American Mathematical Society, (2011).
- [5] Colding, T.H., Minicozzi II, W.P.: Generic mean curvature flow I; generic singularities. Annals of Mathematics 175, no. 2, 755-833 (2012).
- [6] Colding, T.H., Minicozzi II, W.P.: Smooth compactness of self-shrinkers. Comment. Math. Helv. 87, 463-475 (2012).
- [7] Colding, T.H., Minicozzi II, W.P.: Estimates for Parametric Elliptic Integrands, International Mathematics Research Notices, no. 6, 291-297 (2002).
- [8] Ecker, K.: Regularity Theory for Mean Curvature Flow, Progress in nonlinear differential equations and their applications, 75, Birkhauser, Boston, (2004).
- [9] Hamilton, R. S.: Harnack estimate for the mean curvature flow. J. Differential Geom. 41, 215-226 (1995).
- [10] Hauswirth, L., Rosenberg H., Spruck, J.: On complete mean curvature  $1/2$  surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ . Comm. Anal. Geom. 16. no. 5, 989-1005 (2008).
- [11] Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differential Geom. 31, no. 1, 285-299 (1990).
- [12] Huisken, G., Sinestrari, C.: Mean curvature flow singularities for mean convex surfaces, Calc. Var. Partial Differential Equations 8, no. 1, 1-14 (1999).
- [13] Huisken G., Sinestrari, C.: Convexity estimates for mean curvature flow and singularities of mean convex surfaces, Acta Math., 183, 45-70, (1999).



- [14] Schoen, R., Simon, L., Yau, S. T.: Curvature estimates for minimal hypersurfaces, Acta Math. 134, no. 3-4, 275-288 (1975).
- [15] Schoen, R.: Estimates for stable minimal surfaces in three-dimensional manifolds. Seminar on minimal submanifolds, Ann. of Math. Stud., 103, Princeton Univ. Press, Princeton, NJ, 111-126 (1983).

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